Multiple scattering between two elastic cylinders and invariants of the time-reversal operator: Theory and experiment

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(Received 18 October 2005; revised 31 May 2006; accepted 1 June 2006)

The decomposition-of-the-time-reversal-operator method is an ultrasonic method based on the analysis of the array response matrix used for detection and characterization. The eigenvalues and the eigenvectors of the time-reversal operator (equivalent to the singular values and the singular vectors of the array response matrix) provide information on the localization and nature of scatterers in the insonified medium. Here, the eigenmodes of the time-reversal operator are studied for two elastic cylinders: The effects of multiple scattering and anisotropic scattering are considered. Analytical expressions for the singular values are established within the isotropic scattering approximation. Then, the comparison with a complete model is presented, putting in evidence the importance of the anisotropy of the scattering. Experiments, carried out at central frequency 1.5 MHz on 0.25 mm diameter nylon and copper cylinders embedded in water, confirm the theory. In particular, the small cylinder limit and the effect of the dominant quadrupolar normal mode of nylon are discussed. © *2006 Acoustical Society of America.* [DOI: 10.1121/1.2217128]

PACS number(s): 43.60.Pt, 43.28.We, 43.20.Fn [DRD] Pages: 875–883

I. INTRODUCTION

The analysis of acoustic scattering is an important tool for imaging and object identification. It has applications among nondestructive evaluation, medical imaging, or underwater acoustics. The decomposition-of-the-time-reversal-operator (DORT) method is an original approach to scattering analysis which has been developed since 1994. It was derived from the theoretical analysis of acoustic time-reversal mirrors used in pulse echo mode. DORT is the French acronym for Décomposition de l'Opérateur de Retournement Temporel. It consists of the determination of the invariants of the time-reversal operator obtained by singular value decomposition of the array response matrix **K**. It was applied to detection and selective focusing through nonhomogeneous media containing multiple targets.[1] It has also been applied to nondestructive evaluation[2] and characterization of a cylindrical shell through the analysis of the circumferential Lamb waves.[3] Besides, the DORT method has shown potential for highly resolved detection in a water waveguide, experimentally[4–6] and theoretically.[7,8] This method is general and applies to all types of linear waves, thus it is also studied for electromagnetic applications.[9,10]

The first study of the invariants of the time-reversal operator for two scatterers was presented in 1996 by Prada *et al.*[11] Considering isotropic scatterers and single scattering, the eigenproblem was solved. Recently, that point of view was used by Lehman and Devaney,[12] to achieve time-reversal imaging. The effect of multiple scattering was first addressed in subwavelength localization experiments, by Prada and Thomas.[13] It was shown that, for closely spaced scatterers, multiple scattering becomes significant and affects the singular values of the array response matrix, however leaving the rank of this matrix unchanged. A rough model assuming isotropic scattering was used but not described in this paper. Recently, Devaney *et al.*[14] provided a theoretical framework that takes into account multiple isotropic scattering to achieve high-resolution time-reversal imaging. In Refs. 11–14, the scattering was always supposed to be isotropic, but recent analysis on elastic spheres[15] and cylinders[16] showed that, even for a single small scatterer, the anisotropy of the scattering leads to multiple singular values and singular vectors. Consequently, anisotropy has to be taken into account for an accurate calculation of the invariants of the time-reversal operator for two elastic cylinders.

The scattering of two parallel elastic cylinders was first described by Twersky in 1952,[17] in terms of multiple scattering between two anisotropic scatterers. The isotropic scattering approximation and small cylinder limit were presented. That model, valuable for a large separation compared to the wavelength, was then completed in various papers. Among them, there was a clear description provided by Young and Bertrand in 1975,[18] which is used in the present paper to calculate the array response matrix.

Here, the effect of both multiple scattering and anisotropic scattering on the singular values of the two elastic cylinders problem is analyzed. First, generalities about the DORT method, the scattering of a single elastic cylinder, and the isotropic scattering approximation are briefly recalled in Sec. II. Then, a complete model for two elastic cylinders is presented in Sec. III, taking into account all significant normal modes of scattering. The rank of the array response matrix **K** is discussed. Then, **K** is written within the isotropic scattering approximation, taking into account the monopolar normal modes. Within that approximation, analytical expressions of the singular values and singular vectors, for two identical cylinders in symmetrical positions, are provided. These expressions bring an overall physical understanding of

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the role of multiple scattering. In order to improve the description, a correction to the isotropic model using the values of back- and sidescattering is proposed. Approximations and the complete model are then discussed. Finally, in Sec. IV, experimental results on 0.25 mm diameter copper and nylon cylinders are presented, and compared to the complete model.

II. GENERALITIES

In this part, some well known results are briefly recalled to set the framework of the analysis. An array of N transmit-receive transducers, used in a time-invariant scattering medium, is characterized at each frequency ω by the array response matrix $\mathbf{K}(\omega)$,[1] the elements of which are the Fourier transform at frequency ω of the $N\times N$ interelement impulse responses. The receive vector $\mathbf{R}(\omega)$ is related to the transmit vector $\mathbf{E}(\omega)$ through the equation $\mathbf{R}(\omega)=\mathbf{K}(\omega)\mathbf{E}(\omega)$. The time-reversal operator ${}^t\mathbf{K}^*\mathbf{K}$ is diagonalizable (the notation * and ^{t} mean complex conjugate and transpose operations), and its eigenvectors can be interpreted as invariants of the time-reversal process. In fact, the eigenvectors of ${}^t\mathbf{K}^*\mathbf{K}$ and $\mathbf{K}{}^t\mathbf{K}^*$ are the singular vectors of the array response matrix $\mathbf{K}$, while the eigenvalues are the square of the singular values of $\mathbf{K}$.[1] Consequently, the DORT method, which consists of the analysis of the invariants of the time-reversal operator, requires the singular value decomposition (SVD) of the array response matrix $\mathbf{K}$. The SVD is written $\mathbf{K}=\mathbf{U}\mathbf{\Sigma}{}^t\mathbf{V}^*$, where $\mathbf{\Sigma}$ is a real positive diagonal matrix of singular values σ_j, $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices—the column of which are the singular vectors $\mathbf{U}_j$ and $\mathbf{V}_j$, with $1\leqslant j\leqslant N$. Thanks to reciprocity, $\mathbf{K}$ is a symmetrical matrix, and it is straightforward to show that $\mathbf{U}_j$ is the conjugate of $\mathbf{V}_j$ multiplied by an undetermined phase term ϕ_j: $\mathbf{U}_j=\mathbf{V}_j^* e^{i\phi_j}$. In the following, for simplicity and uniqueness, $\mathbf{U}_j$ is chosen equal to $\mathbf{V}_j^*$ (i.e., $\phi_j=0$). In that case, the SVD is written $\mathbf{K}=\mathbf{U}\mathbf{\Sigma}{}^t\mathbf{U}$.

A. The case of a single elastic cylinder

A single elastic cylinder (number 1) of radius a_1, perpendicular to a linear array of transducers, is placed at a distance $F\gg a_1$ from the array plane and at a distance dy_1 from the array axis (Fig. 1). The transducers are supposed to be long rectangles so that the problem can be considered as two dimensional (2D). The response from transducer number j to the scatterer is written H_{1j}. The Green function is approximated by the 2D far-field Hankel function of the first kind $H_0^{(1)}(k_0 r_{1j})$. Taking into account the aperture function of the transducer, $O_{1j}=\mathrm{sinc}[A(y_j-dy_1)/r_{1j}]$, the response is written as

$$H_{1j}=O_{1j}\sqrt{\frac{2}{i\pi k_0 r_{1j}}}e^{ik_o r_{1j}}, \quad 1\leqslant j\leqslant N, \tag{1}$$

where k_0 is the wave number in water and r_{1j} is the distance between the jth transducer and cylinder 1, $r_{1j}=\sqrt{F^2+(y_j-dy_1)^2}$. The $1\times N$ vector of components H_{1j}, denoted $\mathbf{H}_1$, describes the propagation from the N transducers to the scatterer. Due to the reciprocity principle, the backpropagation, from the scatterer to the transducers, is described by ${}^t\mathbf{H}$. The scattered pressure by an elastic cylinder is a sum of normal modes.[21] Hence, the expression of the array response matrix $\mathbf{K}$ is ${}^t\mathbf{H}_1\widetilde{\mathbf{C}}_1\mathbf{H}_1$, where the $N\times N$ scattering matrix $\widetilde{\mathbf{C}}_1$ has the following components:[16]

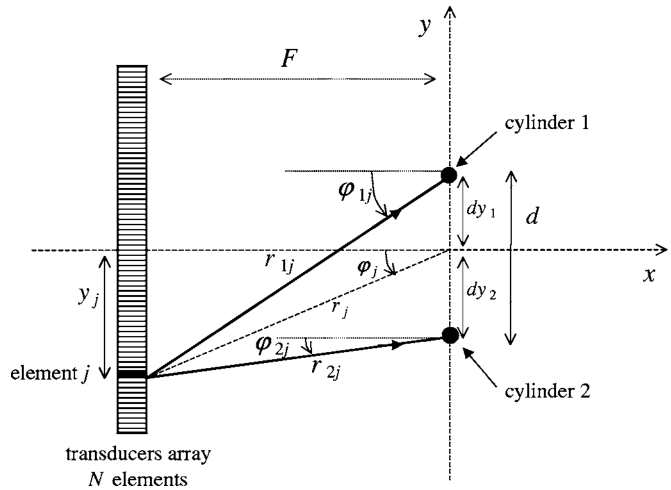

FIG. 1. Geometry of the experiment: Two elastic cylinders, located at positions dy_1 and dy_2, are placed at a distance F from the array. The distance between the cylinders is denoted by d. Every transducer position is denoted by y_j (with j ranging from 1 to N). The problem is considered as 2D.

$$\widetilde{C}_{1ij}=\sum_{n=-\infty}^{\infty} R_{1,n}(-1)^n e^{in\varphi_{1ij}}. \tag{2}$$

The terms $R_{1,n}$ are the scattering coefficients given by Flax *et al.*[21] They are functions of the density ρ_1, the radius a_1, and the transverse and longitudinal wave speeds (c_{T1} and c_{L1}) of cylinder 1, and the density ρ_0 and wave speed c_0 of the surrounding fluid. The angle between emission and reception directions is φ_{1ij} equal to $\varphi_{1j}-\varphi_{1i}$ (Fig. 1). That sum is formally infinite, though the terms for which $n\gg k_0 a$ are negligible. If m denotes the highest normal mode order taken into account, the rank of $\widetilde{\mathbf{C}}_1$ is less or equal to $2m+1$, the number of normal modes. Within the small cylinder limit ($k_0 a<0.5$) $m=1$, so that the rank is 3.[16] Furthermore, the rank of $\mathbf{K}$ is equal to the rank of $\widetilde{\mathbf{C}}_1$, since it is less than N. The singular vectors are combinations of the projections of the normal modes onto the array.[16]

B. Isotropic scattering approximation

Within the isotropic approximation, the scattered pressure expression is reduced to one normal mode: The monopolar one.[16] The scattering matrix $\widetilde{\mathbf{C}}_1$ reduces to the complex term $R_{1,0}$. Denote by $R_{1,0}=|R_{1,0}|e^{i\phi_{1,0}}$, with $|R_{1,0}|$ as the modulus of the scattering coefficient and $\phi_{1,0}$ as the scattering phase shift. Let $\|\mathbf{H}_1\|$ be the norm of the vector $\mathbf{H}_1$. The expression of the array response matrix $\mathbf{K}$ is then $\mathbf{K}={}^t\mathbf{H}_1 R_{1,0}\mathbf{H}_1$. The matrix $\mathbf{K}$ is rank 1 by construction, so that the SVD is written as $\mathbf{K}=\widetilde{\mathbf{U}}_1\widetilde{\sigma}_1{}^t\widetilde{\mathbf{U}}_1$, where $\widetilde{\sigma}_1$ is the real positive singular value and $\widetilde{\mathbf{U}}_1$ is the normalized singular vector. The singular value is expressed as

$$\widetilde{\sigma}_1=|R_{1,0}|\,\|\mathbf{H}_1\|^2, \tag{3}$$

and the $N\times 1$ singular vector is

$$\widetilde{\mathbf{U}}_1 = \frac{{}^t\mathbf{H}_1 e^{i\phi_{1,0}/2}}{\|\mathbf{H}_1\|}. \quad (4)$$

With that convention, the phase of a singular vector corresponds to the phase shift due to the propagation from the array to the scatterer plus one-half of the scattering phase shift.

III. THEORY

Two elastic cylinders, noted 1 and 2, are placed at a distance F from the array (Fig. 1). The distance between the cylinder axes is denoted by d. To apprehend the effect of multiple scattering and anisotropic scattering on the singular values, the array response matrix is first expressed using a complete model, then different approximations are proposed.

A. General case: Two elastic cylinders complete model

The expression of the array response matrix **K** is derived from Eq. (21) of Young and Bertrand.[18] The vectors $\mathbf{H}_1$ and $\mathbf{H}_2$ are defined as in Sec. II, as the responses from the array to the centers of the cylinders [Eq. (1)]. **K** is written as the sum of four terms: Two terms with the same propagation vectors $\mathbf{H}_j$ ($j=1, 2$), and two terms coupling vectors $\mathbf{H}_1$ and $\mathbf{H}_2$:

$$\mathbf{K} = {}^t\mathbf{H}_1\mathbf{C}_1\mathbf{H}_1 + {}^t\mathbf{H}_2\mathbf{C}_2\mathbf{H}_2 + {}^t\mathbf{H}_1\mathbf{C}_{1\text{-}2}\mathbf{H}_2 + {}^t\mathbf{H}_2\mathbf{C}_{2\text{-}1}\mathbf{H}_1. \quad (5)$$

For a distance d, small compared to F, the far-field approximation made in Ref. 18 leads to $\varphi_{1j} \approx \varphi_{2j} \approx \varphi_j$, with $1 \leqslant j \leqslant N$, where φ_j is the angle with the reference axis (Fig. 1). The elements of the $N \times N$ scattering matrices **C** are then written as

$$C_{1ij} = \sum_{n=-\infty}^{+\infty} i^{-n} W_n^- e^{in\varphi_{ij}}, \quad (6a)$$

$$C_{2ij} = \sum_{n=-\infty}^{+\infty} i^{n} W_{-n}^+ e^{in\varphi_{ij}}, \quad (6b)$$

$$C_{1-2ij} = \sum_{n=-\infty}^{+\infty} i^{n} X_{-n}^- e^{in\varphi_{ij}}, \quad (6c)$$

$$C_{2-1ij} = \sum_{n=-\infty}^{+\infty} i^{-n} X_{n}^+ e^{in\varphi_{ij}}, \quad (6d)$$

where coefficients $X_n^\pm$ and $W_n^\pm$ are functions of the scattering coefficients of each cylinder $R_{1,n}$, $R_{2,n}$ and of the Hankel functions $H_n^{(1)}(k_0 d)$ describing the propagation between the two cylinders. As in the case of a single elastic cylinder, m denotes the highest normal mode order taken into account. In Appendix A, it is shown that the rank of **K** is less than or equal to $2(2m+1)$, which is the sum of the rank of $\widetilde{\mathbf{C}}_1$ and $\widetilde{\mathbf{C}}_2$ (Sec. II A). This means that multiple scattering leaves the rank unchanged. The expressions for $X_n^\pm$ and $W_n^\pm$ are given in Ref. 18 for two perfectly rigid cylinders. In order to compare with experimental results in Sec. IV, the coefficients of **K** using the exact scattering coefficients of each copper or nylon cylinder will be calculated, as described by Decanini *et al.*[19] Before, in Sec. III B, the analytic expression of the array response matrix using the isotropic scattering approximation as in Twersky[17] is given.

B. Isotropic scattering approximation

In this section, the singular values of the two cylinder problem are studied; writing the array response matrix **K** within the isotropic scattering approximation. Then, the SVD equation is projected in order to reduce the N-dimensional problem to a 2D problem. Finally, the analytic expressions of the singular values for two identical cylinders in symmetrical positions are established and discussed.

1. Expressions of the array response matrix K

The isotropic scattering approximation means that the scattering sum C_{ij} [Eq. (6)] is limited to monopolar terms, $n=0$. The coefficients $X_0^\pm$ and $W_0^\pm$ depend on the scattering coefficients $R_{j,0}$ ($j=1, 2$) and on the Hankel function of the first kind $H_0^{(1)}(k_0 d)$, denoted h. Their expressions are

$$W_0^- = \frac{R_{1,0}}{1 - R_{1,0}R_{2,0}h^2}, \quad (7a)$$

$$W_0^+ = \frac{R_{2,0}}{1 - R_{1,0}R_{2,0}h^2}, \quad (7b)$$

$$X_0^- = X_0^+ = \frac{R_{1,0}R_{2,0}h}{1 - R_{1,0}R_{2,0}h^2}. \quad (7c)$$

Using these expressions, the array response matrix **K** can be written as

$$\mathbf{K} = (\mathbf{K}^{(1)} + \mathbf{K}^{(2)})\frac{1}{1 - R_{1,0}R_{2,0}h^2}. \quad (8)$$

An interpretation of each term in Eq. (8) is now given. First, single scattering is considered, and only the direct scattering between the array and the cylinders are taken into account.[11,13] This is the distorted wave Born approximation described in Refs. 12 and 14. In that case, the array response matrix, noted $\mathbf{K}^{(1)}$ for a single scattering contribution, is the sum of two array response matrices [Fig. 2(a)], each one corresponding to a single isotropic cylinder as described in Sec. II B:

$$\mathbf{K}^{(1)} = {}^t\mathbf{H}_1 R_{1,0}\mathbf{H}_1 + {}^t\mathbf{H}_2 R_{2,0}\mathbf{H}_2. \quad (9)$$

Likewise, the double scattering corresponds to the paths described in Fig. 2(b). Thus, the double scattering contribution noted $\mathbf{K}^{(2)}$ is written as

$$\mathbf{K}^{(2)} = {}^t\mathbf{H}_2 R_{2,0} h R_{1,0}\mathbf{H}_1 + {}^t\mathbf{H}_1 R_{1,0} h R_{2,0}\mathbf{H}_2. \quad (10)$$

Multiple scattering between the cylinders is also taken into account. Two cases are distinguished depending on the parity of the number of scatterings. If that number is odd, the propagation vectors $\mathbf{H}_j$ ($j=1, 2$), from the array and back to the array, are the same [Fig. 3(a)]. On the contrary, in the even case, the propagation vectors are different [Fig. 3(b)].

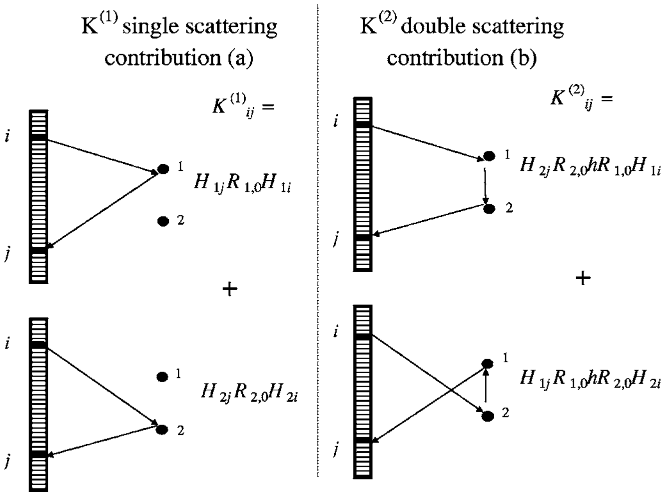

FIG. 2. Expressions of the single scattering contribution $\mathbf{K}^{(1)}$ (a), and double scattering contribution $\mathbf{K}^{(2)}$ (b). A element $K_{ij}^{(1)}$ is a sum of two terms, each one corresponds to a single scattering on the cylinder 1 ($H_{1j}R_{1,0}H_{1i}$) or 2 ($H_{2j}R_{2,0}H_{2i}$). A element $K_{ij}^{(2)}$ is also a sum of two terms. The first one corresponds to the scattering on the cylinder 1 then on the cylinder 2: $H_{2j}R_{2,0}hR_{1,0}H_{1i}$. The second one corresponds to the scattering on the cylinder 2, then the cylinder 1: $H_{1j}R_{1,0}hR_{2,0}H_{2i}$. The propagation term between the two scatterers is denoted h.

The term $R_{1,0}R_{2,0}h^2$ corresponds to the weight of a round trip between the cylinders with two scatterings. Hence, the third scattering order contribution $\mathbf{K}^{(3)}$ is equal to $R_{1,0}R_{2,0}h^2\mathbf{K}^{(1)}$. Similarly, $\mathbf{K}^{(4)}=R_{1,0}R_{2,0}h^2\mathbf{K}^{(2)}$, $\mathbf{K}^{(5)}=(R_{1,0}R_{2,0}h^2)^2\mathbf{K}^{(1)}$, and so on. Therefore, the asymptotic value of the array response matrix $\mathbf{K}$ corresponds to Eq. (8): The sum of the single and double scattering contributions $\mathbf{K}^{(1)}+\mathbf{K}^{(2)}$ multiplied by the asymptotic value of the geometric sum of the ratio $R_{1,0}R_{2,0}h^2$.

2. Projected array response matrix S

The single scattering contribution $\mathbf{K}^{(1)}$ [Eq. (9)] is expressed as

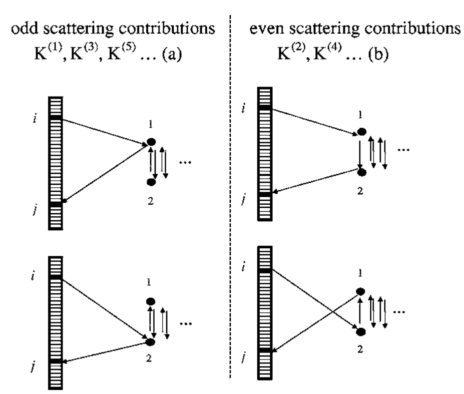

FIG. 3. Expression of the nth scattering order contribution $\mathbf{K}^{(n)}$ as function of the parity of the number of scatterings. The term corresponding to two propagations between the cylinders and two scatterings is equal to $R_{1,0}R_{2,0}h^2$. For odd contribution (a), $\mathbf{K}^{(2n+1)}$ is equal to $(R_{1,0}R_{2,0}h^2)^n\mathbf{K}^{(1)}$. For even contribution (b), $\mathbf{K}^{(2n)}$ is equal to $(R_{1,0}R_{2,0}h^2)^n\mathbf{K}^{(2)}$.

$$\mathbf{K}^{(1)} = \widetilde{\mathbf{U}}_1\widetilde{\sigma}_1{}^t\widetilde{\mathbf{U}}_1 + \widetilde{\mathbf{U}}_2\widetilde{\sigma}_2{}^t\widetilde{\mathbf{U}}_2, \tag{11}$$

where $\widetilde{\sigma}_j$ and $\widetilde{\mathbf{U}}_j$ ($j=1, 2$) are the singular values and vectors if each scatterer is alone [Eqs. (3) and (4)]. In general, they are not singular vectors anymore as $\widetilde{\mathbf{U}}_1$ and $\widetilde{\mathbf{U}}_2$ are not orthogonal. By analogy $\widetilde{\sigma}_{12}$ denotes the weight of the double scattering interaction:

$$\widetilde{\sigma}_{12} = he^{i/2(\phi_{1,0}+\phi_{2,0})}|R_{1,0}|\,|R_{2,0}|\,\|\mathbf{H}_1\|\,\|\mathbf{H}_2\|. \tag{12}$$

The $\widetilde{\sigma}_{12}$ term is complex, whereas the singular values $\widetilde{\sigma}_j$ are real positive. Thus, the double scattering contribution $\mathbf{K}^{(2)}$ [Eq. (10)] is written as

$$\mathbf{K}^{(2)} = \widetilde{\mathbf{U}}_1\widetilde{\sigma}_{12}{}^t\widetilde{\mathbf{U}}_2 + \widetilde{\mathbf{U}}_2\widetilde{\sigma}_{12}{}^t\widetilde{\mathbf{U}}_1. \tag{13}$$

The matrix $\mathbf{K}$ is a linear combination of $\widetilde{\mathbf{U}}_1$ and $\widetilde{\mathbf{U}}_2$ [Eq. (8)], hence it is rank 2. This is in agreement with Devaney *et al.*[14] Thus, the singular value decomposition of $\mathbf{K}$ is written as

$$\mathbf{K} = \mathbf{U}_1\sigma_1{}^t\mathbf{U}_1 + \mathbf{U}_2\sigma_2{}^t\mathbf{U}_2, \tag{14}$$

where σ_j and $\mathbf{U}_j$ ($j=1, 2$) are the singular values and vectors of the two cylinder problem. The singular vector $\mathbf{U}_j$ is expressed as a linear combination of the vectors $\widetilde{\mathbf{U}}_1$ and $\widetilde{\mathbf{U}}_2$

$$\mathbf{U}_j = \chi_{1j}\widetilde{\mathbf{U}}_1 + \chi_{2j}\widetilde{\mathbf{U}}_2, \quad j=1,2. \tag{15}$$

The χ_{ij} terms are complex. The term $\boldsymbol{\chi}_j$ denotes the 2×1 vector containing the term χ_{1j} and χ_{2j}. Expressing $\mathbf{K}$ and $\mathbf{U}_j$ as functions of $\widetilde{\mathbf{U}}_j$, the N-dimensional problem can be reduced to a 2D problem. The relation $\mathbf{K}\mathbf{U}_j^*=\sigma_j\mathbf{U}_j$ is expressed as $\mathbf{S}\boldsymbol{\chi}_j^*=\sigma_j\boldsymbol{\chi}_j$, where $\mathbf{S}$ is the a matrix of dimension 2 and corresponds to the projection of the array response matrix on the subspace Span $\{\widetilde{\mathbf{U}}_1,\widetilde{\mathbf{U}}_2\}$ (Appendix B). Accordingly, it leads to $\mathbf{S}\mathbf{S}^*\boldsymbol{\chi}_j=\sigma_j^2\boldsymbol{\chi}_j$. That is to say σ_j^2 and $\boldsymbol{\chi}_j$ are, respectively, the eigenvalues and eigenvectors of $\mathbf{S}\mathbf{S}^*$, which corresponds to the projection of the time-reversal operator $\mathbf{K}\mathbf{K}^*$ on the subspace Span $\{\widetilde{\mathbf{U}}_1,\widetilde{\mathbf{U}}_2\}$.

3. Resolution for two identical cylinders in symmetrical positions

The calculations of the singular values and vectors for two identical cylinders in symmetrical positions are explained in Appendix C. Denote by w_{12} as the hermitian scalar cross product ${}^t\widetilde{\mathbf{U}}_1^*.\widetilde{\mathbf{U}}_2$, which is real in the symmetrical case. Denote by $\widetilde{\sigma}$ as the singular value for a single cylinder of monopolar term R_0. In that case, the singular values are

$$\sigma_+ = \widetilde{\sigma}(1+w_{12})\left|\frac{1}{1-R_0h}\right|, \tag{16a}$$

$$\sigma_- = \widetilde{\sigma}(1-w_{12})\left|\frac{1}{1+R_0h}\right|, \tag{16b}$$

associated with the singular vectors

$$\mathbf{U}_+ \propto (\widetilde{\mathbf{U}}_1+\widetilde{\mathbf{U}}_2)/\|\widetilde{\mathbf{U}}_1+\widetilde{\mathbf{U}}_2\|, \tag{17a}$$

$$\mathbf{U}_- \propto (\widetilde{\mathbf{U}}_1-\widetilde{\mathbf{U}}_2)/\|\widetilde{\mathbf{U}}_1-\widetilde{\mathbf{U}}_2\|. \tag{17b}$$

C. Back- and sidescattering approximation

In this section, a correction to the isotropic scattering approximation using only two values of the anisotropic scattering values is proposed: The backscattering amplitude ($\varphi = 0°$) and the sidescattering amplitude ($\varphi = 90°$). Denote by C_0 and C_{90}, the values of the scattering amplitude at those angles [Eq. (2)]. That approximation assumes that the back- and sidescattering are constant within the array aperture; hence, the singular values calculation is similar to the isotropic scattering calculation (Appendix D). The singular values are then

$$\sigma_+ = \tilde{\sigma}(1 + w_{12})\left| 1 + h\frac{C_{90}^2}{C_0}\frac{1}{1 - C_0 h}\right|, \tag{18a}$$

$$\sigma_- = \tilde{\sigma}(1 - w_{12})\left| 1 - h\frac{C_{90}^2}{C_0}\frac{1}{1 + C_0 h}\right|. \tag{18b}$$

To provide an analytical expressions for C_0 and C_{90}, the small cylinder limit is used, valid for $k_0 a < 0.5$, described by Minonzio *et al.*[16] and Twersky.[17] In that case, the scattering is the sum of two normal modes (monopole and dipole). It depends on the compressibility contrast α, the density contrast β and on a scattering coefficient c, which are written[20]

$$\alpha = 1 - \frac{\rho_0 c_0^2}{\rho_1 (c_L^2 - c_T^2)}, \tag{19a}$$

$$\beta = 2\frac{\rho_1 - \rho_0}{\rho_1 + \rho_0}, \tag{19b}$$

$$c = -i\pi k_0^2 a^2/4. \tag{19c}$$

The weight of the monopolar mode is $R_0 = \alpha c$, and the weight of the dipolar mode is $-2R_1 = \beta c$. Thus, the backscattering coefficient is $C_0 = (\alpha + \beta)c$, and the sidescattering coefficient is $C_{90} = \alpha c$. In both cases, the scattering phase shift is $\phi = -\pi/2$.

D. Comparison of the three models

In Fig. 4, results given by the three models are compared: Isotropic scattering approximation [Eq. (16)], back- and sidescattering approximation [Eq. (18)], and the complete model. The normalized singular values are shown for two identical cylinders, at a single frequency, versus the distance d between the two cylinders. The diameters are equal to 0.25 mm and the frequency is 1.5 MHz, hence $k_0 a = 0.8$. Separation d ranges from contact ($d = 2a$) to 3 mm. Figure 4(a) shows the copper case, and Fig. 4(b) shows the nylon case. Physical parameters taken into account for calculations are given in Table I.

The isotropic and the back- and sidescattering approximations give the same expression for the singular values for single scattering, keeping only the first-order term: $\sigma_\pm^{(1)} = \tilde{\sigma}(1 \pm w_{12})$. As in Sec. III B, superscript (1) is used for single scattering. That expression is equivalent to Eq. (25) of Prada *et al.*[11] Single scattering singular values are shown as a dashed line (Fig. 4). If d is small compared to the resolution cell, the nonresolved case, w_{12} is close to 1: $\sigma_+^{(1)} \approx 2\tilde{\sigma}$ and $\sigma_-^{(1)} \approx 0$. On the contrary, if d is large compared to the resolution cell, the well-resolved case, w_{12} is close to 0: $\sigma_+^{(1)} \approx \sigma_-^{(1)} \approx \tilde{\sigma}$.[11–13] Note that the SVD gives the singular values in order of importance, i.e., $\sigma_1 \geqslant \sigma_2$. On the contrary, the previous equation gives $\sigma_+^{(1)}$ inferior to $\sigma_-^{(1)}$ when w_{12} is negative.

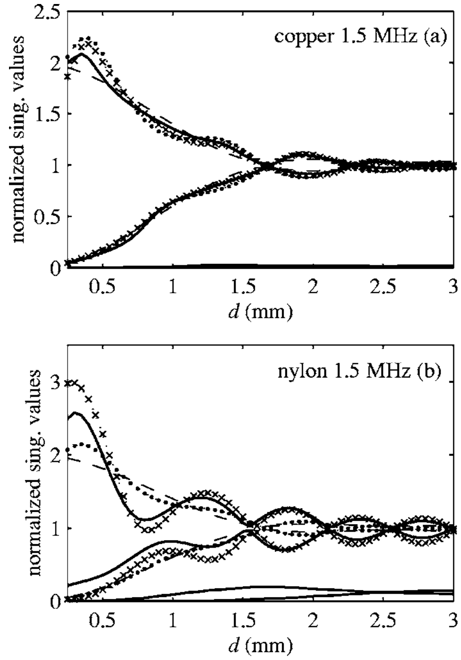

FIG. 4. Normalized singular values versus separation d, for two 0.25 mm diameter cylinders, at 1.5 MHz, copper (a) and nylon (b): Single scattering (dashed line), isotropic scattering approximation (•), back- and sidescattering approximation (×) and complete model (solid line).

For the three models, the singular values $\sigma_\pm$ present oscillations compared with the single scattering singular value $\sigma_\pm^{(1)}$ (Fig. 4). For $d < 0.6$ and $d > 1.1$, σ_+ is larger than $\sigma_+^{(1)}$, whereas for $0.6 < d < 1.1$, σ_+ is smaller than $\sigma_+^{(1)}$. The second singular value σ_- presents opposite variations compared with $\sigma_-^{(1)}$. It is possible to explain these oscillations looking at the expression for the singular values [Eqs. (16) and (18)]. In both cases, the singular values can be written as a second-order Taylor expansion:

$$\sigma_\pm \approx \tilde{\sigma}(1 \pm w_{12})\{1 \pm |x||h|\cos(k_0 d - \pi/4 + \phi_x) + o(h^2)\}, \tag{20}$$

where x is equal to R_0 or C_{90}^2/C_0 and ϕ_x is the phase of x. The phase of h is equal to $k_0 d - \pi/4$, because h reduces to $e^{ik_0 d}\sqrt{2/i\pi k_0 d}$ for large arguments (valid for $k_0 d > 2$ or $d > 0.3\lambda$). With $\phi_x = -\pi/2$, the oscillations given by the Taylor expansion are in good agreement with the complete model singular values shown in Fig. 4. The two first zeros of the cosine correspond to $k_0 d = 5\pi/4$ and $9\pi/4$ ($d = 0.6$

TABLE I. Physical parameters of copper and nylon.

	ρ (g cm^{-3})	c_L (mm μs^{-1})	c_T (mm μs^{-1})	α	β
Copper	8.9	5.0	2.3	0.99	1.6
Nylon	1.15	2.5	1.0	0.62	0.14
Water	1	1.48			

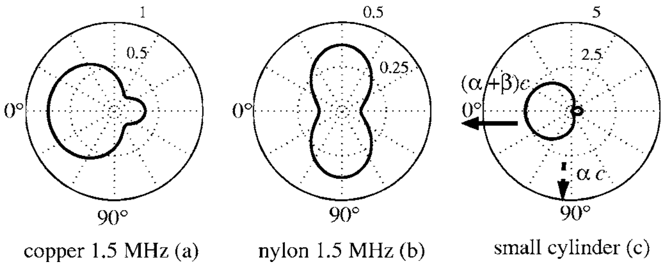

FIG. 5. Scattering patterns $|\tilde{C}(\varphi)|$: 0.25 mm diameter copper (a), nylon (b) at 1.5 MHz and small cylinder limit (c). Patterns (a) and (b) have been calculated with Eq. (2). Pattern (c) is equal to $|\alpha+\beta\cos(\varphi)|$, where α and β are the compressibility and density contrasts of copper (Table I).

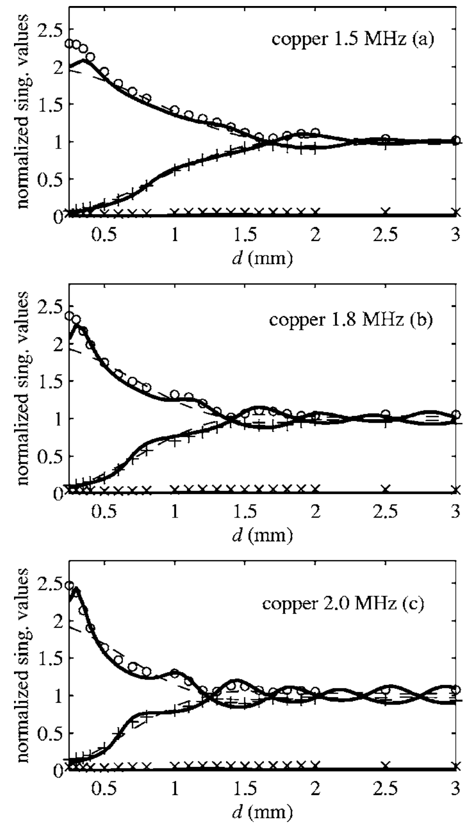

FIG. 6. Two 0.25 mm diameter copper cylinders: Experimental and complete model normalized singular values versus the separation d. Three frequencies are shown: 1.5 MHZ (a), 1.8 MHz (b), and 2 MHz (c), single scattering (dashed line), complete model (solid line), and experimental values $(\mathrm{o},+,\times)$.

and 1.1 mm). Likewise, the extrema are located at $k_0 d = 3\pi/4$ and $7\pi/4$ (d=0.4 and 0.9 mm). Accordingly, the small cylinder limit scattering phase shift $(-\pi/2)$ is correct in that case [Eq. (19c)]. The cosine positive domain corresponds to constructive inferences between single and multiple scattering. On the contrary, the negative domain corresponds to destructive interferences.

However, the amplitude of the singular values given by the isotropic scattering approximation (•) differs from those given by the complete model (solid line). That difference can be explained by the scattering patterns shown in Fig. 5(a) for copper and Fig. 5(b) for nylon. For copper, $|C_{90}|$ is less than $|C_0|$. This is why the complete model oscillations are smaller than the oscillations given by the isotropic scattering approximation [Fig. 4(a)]. On the contrary, for nylon, $|C_{90}|$ is larger than $|C_0|$; so that the complete model oscillations are larger than the isotropic ones [Fig. 4(b)]. The back- and sidescattering approximation $(\times)$ compensates for part of that difference, and agrees really well for large values of d. Furthermore, for metals, as the quadrupolar term R_2 is small, for $k_0 a < 1$, the small cylinder limit [Eq. (19)] is a good approximation. The small cylinder limit scattering pattern [Fig. 5(c)] is close to the copper one [Fig. 5(a)].

Furthermore for nylon, the complete model does not show similar oscillations for the lower pair of singular values σ_3 and σ_4. In the copper case, these singular values are due to the antisymmetric dipolar normal modes, which are maximum for 90° scattering angles, as described by Minonzio *et al.*[16] On the contrary, for nylon, the second pair of singular values seems to be weakly affected by multiple scattering. In that case, these singular values are due to the antisymmetric quadrupolar normal modes.[16] Actually, the nylon dipolar mode is small because of the weak density contrast. For 90° scattering angles, the antisymmetric quadrupolar modes are null, so that for nylon the effect of multiple scattering is weak for those singular values.

IV. EXPERIMENTAL RESULTS AND DISCUSSIONS

A. Experimental setup

In a water tank a 96 element linear array—with 1.5 MHz central frequency and 0.5 mm pitch—is used. The two cylinders are identical, but their positions are not symmetrical. One cylinder is fixed, whereas the second one is connected to a motor. The distance d between the two cylinders is incremented from contact ($d=2a$) to 3 mm. The distance from the array is F=50 mm. Experiments have been carried out for two materials: Copper and nylon. In both cases, the diameters were taken equal to 0.25 mm. With that array, the parameter A of the aperture function does not depend on frequency [Eq. (1)]. It has been experimentally measured and is equal to 1.6. A reception level correction law is used as described by Minonzio *et al.*[16] As the wave backscattered by such small objects is very weak, the Hadamard-Walsh basis with chirps is used to acquire the array response matrix so as to optimize the signal-to-noise ratio in the whole bandwidth (1–2 MHz).[8]

B. Two copper cylinders

The first experiment was carried out on two 0.25 mm diameter copper cylinders. Physical parameters, given in Table I, are taken into account for the complete model calculations. Figures 6(a)–6(c) show the normalized singular values versus the distance for three frequencies, 1.5, 1.8, and 2 MHz. There is good agreement between experimental and theoretical values (complete model). For copper, as already observed in Sec. III D, the small cylinder limit gives good results. Therefore, Eqs. (18) and (19) are sufficient to describe the multiple scattering between two small metallic cylinders. Experimental results are similar to those presented by Prada and Thomas;[13] however, the multiple scattering model was not described.

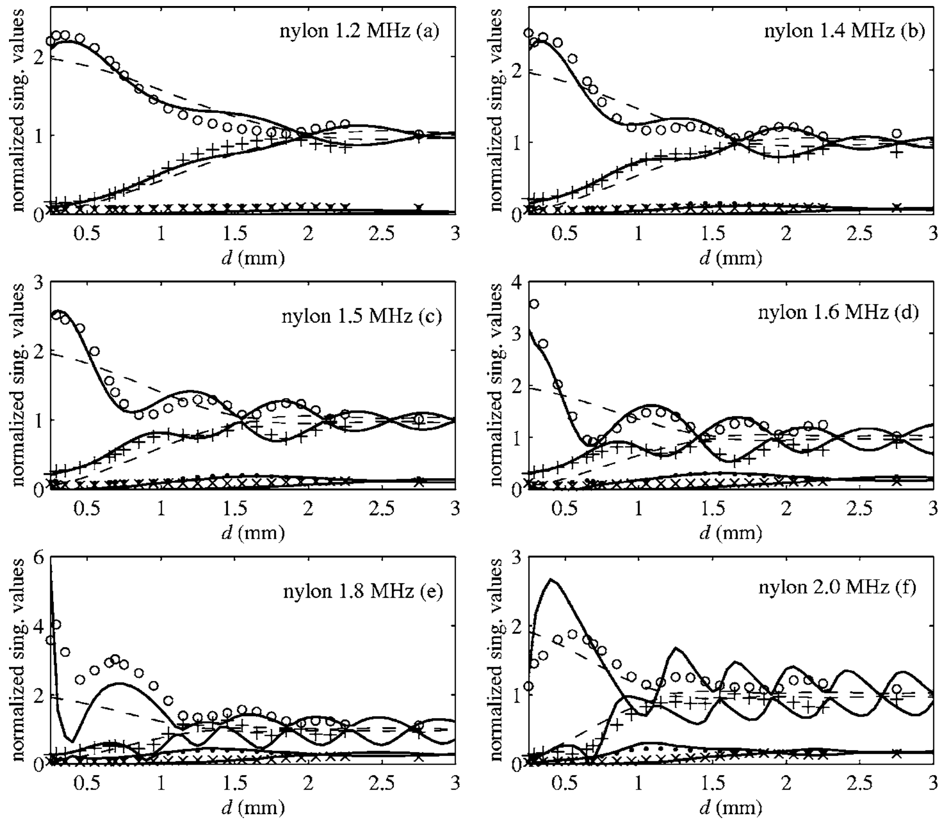

FIG. 7. Two 0.25 mm diameter nylon cylinders: Experimental and complete model normalized singular values versus the separation d. Six frequencies, from 1.2 to 2 MHz are shown, single scattering (dashed line), complete model (solid line), and experimental values $(\circ, +, \bullet, \times)$. The singular values $\boldsymbol{\sigma}_3$ $(\bullet)$ and $\boldsymbol{\sigma}_4$ $(\times)$ are weakly affected by multiple scattering.

C. Two nylon cylinders

The second experiment was carried out on two 0.25 mm diameter nylon cylinders. Physical parameters, given in Table I, are taken into account for the complete model calculations. Figures 7(a)–7(f) show the singular values versus the distance for six frequencies between 1.2 and 2 MHz. There is a good agreement between experimental and theoretical values (complete model). The frequency 2 MHz corresponds to the peak of the quadrupolar term R_2 described in Ref. 16. For that frequency, the scattering pattern is a quadrupole with a scattering phase shift equal to $-\pi$, instead of $-\pi/2$, for the small cylinder limit. Thus the maximum interaction appears for $k_0 d$ equal to $5\pi/4$ (d=0.5 mm), instead of $3\pi/4$. It clearly appears that the small cylinder limit, which does not take the quadrupolar mode into account, is not valid for the nylon cylinders.

Furthermore, the lower-order singular values σ_3 and σ_4 are not negligible, as in the copper case. They are clearly measured for frequencies beyond 1.5 MHz [Figs. 7(c)–7(f)]. As noticed in Sec. III D, the effect of multiple scattering seems to be small for those singular values.

V. CONCLUSION

In order to describe how multiple scattering between two elastic cylinders affects the singular values of the interelement array-response matrix $\mathbf{K}$, three points of view have been proposed: Isotropic scattering, back- and sidescattering, and a complete model. The rank of $\mathbf{K}$ was specified as a function of the number of normal modes taken into account: All the significant ones for the complete model, only the monopolar ones for the isotropic scattering approximation. Within that approximation, for two identical cylinders in symmetrical position, analytical expressions for the singular values $\sigma_\pm$ have been given [Eq. (16)] which, explain overall the oscillations given by the complete model. A correction to the isotropic model is proposed using the back- and sidescattering amplitudes [Eq. (18)]: It compensates for part of the difference between isotropic and complete models. Within the small cylinder limit, back- and sidescattering amplitudes are expressed as a function of the compressibility and density contrasts α and β [Eq. (19)]. For metal cylinders, for $k_0 a < 1$, that expression is sufficient to describe the multiple scattering problem. Experimental results confirm the validity of the model for 0.25 mm diameters nylon and copper cylinders, for frequencies between 1.2 and 2 MHz. The effect of the interaction between the two cylinders is clearly shown. For nylon, the two following singular values σ_3 and σ_4 are weakly affected by multiple scattering because of the predominance of the quadrupolar normal mode.

APPENDIX A: RANK OF K

Let us denote by m the highest normal mode order taken into account in Eq. (6). Let us define four $(2m+1)\times(2m+1)$ diagonal square matrices $\mathbf{W}^{\pm}$ and $\mathbf{X}^{\pm}$:

$$\mathbf{W}^{-} = \mathrm{diag}(i^{-n} W_n^{-}), \tag{A1a}$$

$$\mathbf{W}^{+} = \mathrm{diag}(i^{n} W_{-n}^{+}), \tag{A1b}$$

$$\mathbf{X}^{-} = \mathrm{diag}(i^{-n} X_n^{-}), \tag{A1c}$$

$$\mathbf{X}^{+} = \mathrm{diag}(i^n X^{-}_{-n}), \qquad \text{(A1d)}$$

with $-m \leqslant n \leqslant m$. Let also define $\mathbf{E}$ as a $(2m+1)\times N$ matrix of coefficients $E_{jn}=e^{in\varphi_j}$. Using $\varphi_{ij}=\varphi_j-\varphi_i$, Eq. (5) can be written as

$$\mathbf{K} = {}^t(\mathbf{EH}_1)\mathbf{W}^{-}\mathbf{E}^{*}\mathbf{H}_1 + {}^t(\mathbf{EH}_2)\mathbf{W}^{+}\mathbf{E}^{*}\mathbf{H}_2 + {}^t(\mathbf{EH}_1)\mathbf{X}^{-}\mathbf{E}^{*}\mathbf{H}_2 + {}^t(\mathbf{EH}_2)\mathbf{X}^{+}\mathbf{E}^{*}\mathbf{H}_1, \qquad \text{(A2)}$$

where $\mathbf{K}$ appears to be the product of three larger matrices, such that

$$\mathbf{K} = {}^t\begin{bmatrix}\mathbf{EH}_1\\ \mathbf{EH}_2\end{bmatrix}\begin{bmatrix}\mathbf{W}^{-} & \mathbf{X}^{-}\\ \mathbf{X}^{+} & \mathbf{W}^{+}\end{bmatrix}\begin{bmatrix}\mathbf{E}^{*}\mathbf{H}_1\\ \mathbf{E}^{*}\mathbf{H}_2\end{bmatrix}. \qquad \text{(A3)}$$

The dimension of the square matrix in the center is $2(2m+1)$, so that the rank of $\mathbf{K}$ is necessarily lower than $2(2m+1)$ since it is less than N.

APPENDIX B: EXPRESSION OF THE PROJECTED ARRAY RESPONSE MATRIX S

For two elastic cylinders, within the isotropic scattering approximation, the matrix $\mathbf{S}$ is expressed as

$$\mathbf{S} = (\tilde{\mathbf{\Sigma}}\mathbf{W}_1 + \tilde{\sigma}_{12}\mathbf{W}_2)\frac{1}{1-R_{1,0}R_{2,0}h^2}, \qquad \text{(B1)}$$

where $\tilde{\mathbf{\Sigma}}$, $\mathbf{W}_1$, and $\mathbf{W}_2$ are 2×2 matrices. $\tilde{\mathbf{\Sigma}}$ is diagonal and contains the single cylinder singular values $\tilde{\sigma}_j$ equal to $|R_{j,0}|\,\|\mathbf{H}_j\|^2$ [Eq. (3)]. Let us denote by w_{ij} $(i, j=1, 2)$ the hermitian scalar product equal to ${}^t\mathbf{U}^{*}.\tilde{\mathbf{U}}_j$. As the $\tilde{\mathbf{U}}_j$ are normalized [Eq. (4)], w_{ii} is equal to 1. In general, that product is complex and w_{ji} is equal to $w_{ij}{}^{*}$. The modulus $|w_{ij}|$ ranges from 0 to 1. It corresponds to the normalized acoustic field on the second scatterer when the field is focused on the first one.[11] The matrices $\mathbf{W}_j$ contain the w_{ij} terms. The matrices are written as

$$\tilde{\mathbf{\Sigma}} = \begin{pmatrix}\tilde{\sigma}_1 & 0\\ 0 & \tilde{\sigma}_2\end{pmatrix}, \qquad \text{(B2)}$$

$$\mathbf{W}_1 = \begin{pmatrix}1 & w_{12}^{*}\\ w_{12} & 1\end{pmatrix},$$

$$\mathbf{W}_2 = \begin{pmatrix}w_{12} & 1\\ 1 & w_{12}^{*}\end{pmatrix}.$$

The matrix product $\tilde{\mathbf{\Sigma}}\mathbf{W}_1$ is the single scattering projected matrix. It corresponds to the matrix $\mathbf{S}'$ presented in the Appendix of Ref. 11.

APPENDIX C: RESOLUTION FOR TWO IDENTICAL CYLINDERS IN SYMMETRICAL POSITIONS

The two cylinders are now considered as identical in a symmetrical geometry with respect to the array axis (x axis, Fig. 1), i.e., $dy_1=-dy_2$. If the cylinders are identical, $R_{1,0}$ is equal to $R_{2,0}$: The weight of the monopolar normal modes is denoted R_0. Furthermore, if the positions are symmetrical, the norms $\|\mathbf{H}_j\|$ are also equal. So the $\tilde{\sigma}_j$ coefficients are equal: They are denoted $\tilde{\sigma}$. Likewise, the weight of the double scattering interaction $\tilde{\sigma}_{12}$ is equal to $R_0 h\tilde{\sigma}$ [Eq. (12)], and the round trip interaction term $R_{1,0}R_{2,0}h^2$ is equal to $(R_0h)^2$. For symmetrical positions, the crossed scalar product w_{12} and w_{21} are real and equal. Thus, the symmetrical case simplifies the matrices product $\mathbf{W}_i\mathbf{W}_j$ $(i, j=1, 2)$ as

$$\mathbf{W}_1^2 = \mathbf{W}_2^2 = \begin{pmatrix}1+w_{12}^2 & 2w_{12}\\ 2w_{12} & 1+w_{12}^2\end{pmatrix}, \qquad \text{(C1)}$$

$$\mathbf{W}_1\mathbf{W}_2 = \mathbf{W}_2\mathbf{W}_1 = \begin{pmatrix}2w_{12} & 1+w_{12}^2\\ 1+w_{12}^2 & 2w_{12}\end{pmatrix}. \qquad \text{(C2)}$$

Therefore, the expression of $\mathbf{SS}^{*}$ is

$$\mathbf{SS}^{*} = \tilde{\sigma}^2[(1+|R_0h|^2)\mathbf{W}_1^2 + (R_0h+(R_0h)^{*})\mathbf{W}_1\mathbf{W}_2]\frac{1}{|1-(R_0h)^2|^2}. \qquad \text{(C3)}$$

The expression of the $\mathbf{SS}^{*}$ matrix is symmetrical, in $\begin{pmatrix}a & b\\ b & a\end{pmatrix}$ form. Thus, the two eigenvalues $\sigma_{\pm}^2$ are written $a+b$ and $a-b$, associated with the eigenvectors $1/\sqrt{2}{}^t(1\ \ 1)$ and $1/\sqrt{2}{}^t(1\ \ -1)$. The singular values of $\mathbf{K}$ [Eq. (16)] are the square roots of the eigenvalues of $\mathbf{SS}^{*}$. The singular vectors are given in Eq. (17).

APPENDIX D: BACK-AND SIDESCATTERING APPROXIMATION

Let us denote by C_0 the value of the backscattering; and C_{90}, the value of the sidescattering [Eq. (2)]: C_0 is equal to $R_0-2R_1+2R_2+\ldots$ and C_{90} to $R_0-2R_2+\ldots$ With those notations, the singular value $\tilde{\sigma}$ is equal to $|C_0|\,\|\mathbf{H}_1\|^2$ for a single scatterer, the weight of the double scattering interaction $\tilde{\sigma}_{12}$ [Eq. (12)] is equal to $\tilde{\sigma}C_{90}^2h/C_0$ and the round trip interaction term is equal to $(C_0h)^2$. Accordingly, the expression of the projected array response matrix $\mathbf{S}$ is

$$\mathbf{S} = \tilde{\boldsymbol{\sigma}}\left(\mathbf{W}_1 + h\frac{C_{90}^2}{C_0}(\mathbf{W}_2 + C_0h\mathbf{W}_1)\frac{1}{1-(C_0h)^2}\right). \qquad \text{(D1)}$$

Calculations are similar to those in Appendix C. The singular vectors are the same as before [Eq. (17)]. The singular values are given in Eq. (18).

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