# Edge resonance and zero group velocity Lamb modes in a free elastic plate

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The local resonances of a free isotropic elastic plate are investigated using laser ultrasonic techniques. Experimental results are interpreted in terms of zero group velocity Lamb modes and edge mode. At a distance from the edge larger than the plate thickness a sharp resonance is observed at the frequency where the group velocity of the first symmetrical Lamb mode vanishes. Close to the edge of the plate, the resonance due to the edge mode dominates. Both zero group velocity and edge resonances appear at the theoretically predicted frequencies. These frequencies do not vary with the distance from the edge of the plate and the transition between the two modes of vibration, at about the plate thickness, is abrupt. Using a laser excitation on the edge, the amplitude profile of the normal displacement at the edge resonance frequency was determined.
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## I. INTRODUCTION

Linear free vibrations of an infinite isotropic elastic plate can be interpreted in terms of Lamb modes. Propagating modes are represented by a set of curves giving the real angular frequency $\omega$ of each symmetric ($S$) and antisymmetric ($A$) mode versus the real wave number $k$, solution of the Rayleigh-Lamb dispersion equation.[1] Figure 1 shows the dispersion curves of the lower order modes for a Duralumin plate of thickness $d=2h$ (longitudinal and transverse wave velocities $V_L=6.40$ km/s and $V_T=3.12$ km/s, respectively). Dimensionless quantities $\Omega=\omega h/V_T$ and $kh$ are used. Low frequency and long wavelength extensional or flexural vibrations can be ascribed to the fundamental symmetric $S_0$ or antisymmetric $A_0$ modes, which exhibit free propagation to zero frequency, respectively. High frequency and long wavelength stretch or shear vibrations occur at the cutoff frequency $f_c$ of higher order Lamb modes, whose frequencies depend on the plate thickness and on the bulk wave velocities of the material.[2] These thickness resonances correspond to an in-phase motion of the whole plate associated to vanishing wave number $k$ in the plane of the plate.

As shown in Fig. 1, the first symmetric Lamb mode $S_1$ exhibits a frequency minimum associated with a zero group velocity (ZGV) at a nonzero value $k_0$ of the wave number. The energy deposited on the plate by a local impact is trapped under the source at this ZGV frequency $f_0$. As predicted by Tolstoy and Usdin,[3] this phenomenon must give rise to a sharp resonance peak and to ringing effects detectable over a long time in the source area. This local vibration of an elastic plate was recently explained in terms of Lamb waves in civil engineering by Gibson and Popovics.[4] At the same time, it was shown that the ZGV resonance is remarkably generated and detected using non-contact laser based ultrasonic techniques.[5]

Another ringing phenomenon was discovered at the boundary of a thick circular disc,[6] at the free end of a cylindrical rod[7] and on the contour of a rectangular plate.[8] In all cases, this symmetric vibration is of a resonant nature. This so-called "edge mode" exists in a narrow band around the resonant frequency and the mechanical displacement is confined in a small region near the free edge. For a plate at the edge resonance frequency, which is lower that the minimum frequency of the $S_1$ branch, only the fundamental symmetric Lamb mode $S_0$ can propagate. However, the change in the direction of propagation implies a change in sign for one stress component but not for the other. Thus, stress free conditions imposed on the plate edge cannot be fulfilled by the superposition of incident and reflected propagating $S_0$ modes alone. By solving the Rayleigh-Lamb equation, for complex wave numbers, Mindlin and Medick discovered two complex branches besides the real branch of the $S_1$ mode. As shown in Fig. 2, these branches emanate from the stationary point (or ZGV point) of the $S_1$ Lamb mode.[9] Onoe pointed out the role of complex modes to satisfy boundary conditions at the plate edge.[8] Following Gazis and Mindlin,[10] the extensional $S_0$ mode present in the interior of the plate, must be coupled with complex branches of the dispersion equation. A similar approach, highlighting the theoretical link between edge and ZGV modes, was developed by McNiven[11] for a cylindrical rod.

In the first experiments on edge modes, the resonance was measured using a contact piezoelectric transducer.[6] A transducer excitation was then used by Le Clezio *et al.*[12] for plates and Ratassepp *et al.*[13] for pipes, while the non-contact detection was achieved with a laser probe. Recently, it has been shown that pulsed laser sources provide an invaluable tool for investigating the free local resonance spectrum of elastic plates or tubes.[14,15]

The objective of this study is to investigate, with a full non contact technique, the existence of edge and ZGV

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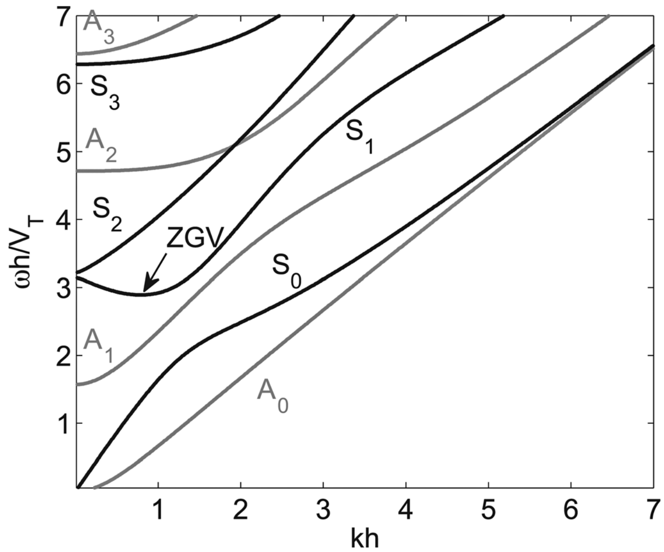


FIG. 1. Dispersion curves of propagating Lamb modes for a Duralumin plate of thickness $d = 2h$.

resonances of an isotropic elastic plate. The paper is organized as the following: the main properties of ZGV Lamb modes and edge resonances are recalled in Sec. II. The local resonances of an elastic plate with a free edge are investigated through laser ultrasonic measurements in Sec. III.

## II. ANALYSIS

As previously indicated, in a thickness resonance the whole plate surface is vibrating in phase, because of the vanishing wave number. Conversely, a ZGV resonance is a local resonance, with a finite wave number. The conditions of existence of zero group velocity Lamb modes in isotropic plates were discussed in a previous paper.[16] These modes appear in a range of Poisson's ratio about the value for which the cut-off frequency curves of modes belonging to the same family intercept. Experimentally, this phenomenon can be observed by detecting the normal displacement at the source point or less than half a wavelength apart from the source. In laser ultrasonic experiments, the absorption of a part of the laser pulse energy generates propagating and non-propagating Lamb modes. Only modes whose energy velocity, i.e., group velocity is zero remain after a long time. The spectrum of the signal exhibits sharp peaks whose frequencies match the ZGV modes. These ZGV resonance frequencies provide the local plate thickness if the longitudinal and transverse wave velocities are known or the material parameters if the plate thickness is known.[15]

The first analysis of the edge resonance, based on the second order Mindlin's approximation, involves only the first two complex branches.[9] If the remarkable increase in the displacement of the plate end is well explained, the agreement with experimental results is only qualitative for the edge resonance frequency. Generally, edge mode is caused by the interferences at the free edge of a plate between the incident and diffracted modes, real and complex ones. Moreover, the edge mode corresponds to a complex valued frequency and its amplitude decay with time at a rate given by the imaginary part of the complex resonance frequency. Recently, a numerical study shows that this resonance frequency becomes real for two values $\nu_1 = 0$ and $\nu_2 = 0.2248$ of the Poisson's ratio $\nu$.[17] Pagneux gives a formula which is a good approximation of this real part $\Omega_R$:

$$\text{Real}[\Omega_R] = 0.652\nu^2 + 0.898\nu + 1.9866. \quad (1)$$

The variations of the real part of normalized resonance frequencies $\Omega = \omega h/V_T$ versus Poisson's ratio are plotted in Fig. 3 for edge, ZGV and thickness modes. This graph confirms

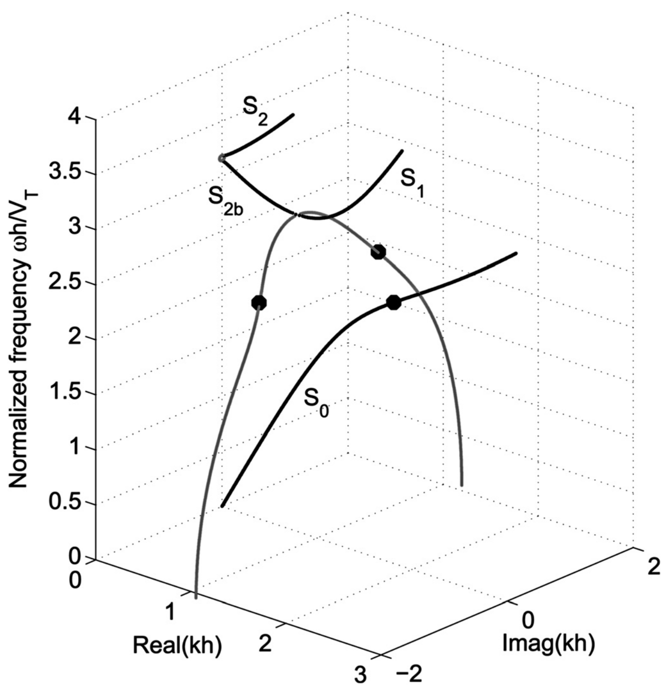


FIG. 2. Dispersion curves of complex Lamb modes for a Duralumin plate. The dots (•) indicate the edge resonance frequency.

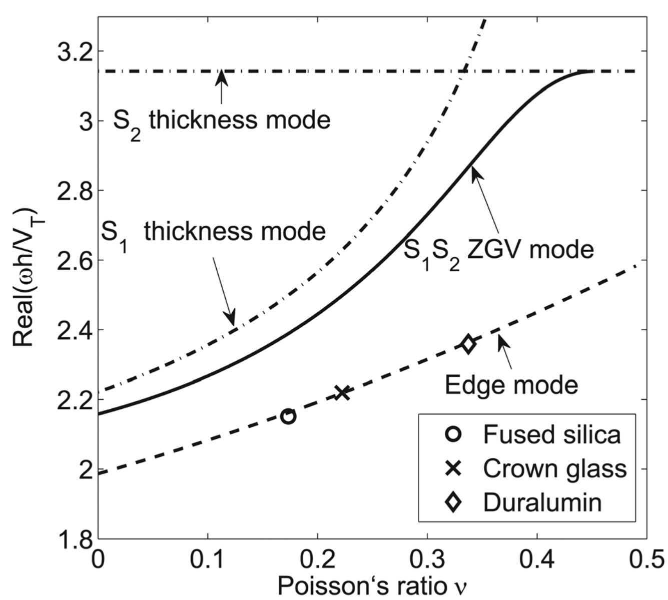


FIG. 3. Real part of normalized resonance frequency $\omega h/V_T$ versus Poisson's ratio ν, for the edge mode (dashed line), the $S_1S_2$-ZGV Lamb mode (solid line) and the thickness modes $S_1$, $S_2$ (dash dot lines). Measured edge resonance frequencies: fused silica (o), crown glass (×), Duralumin (◊).

that for all usual materials, the edge resonance frequency is lower that the ZGV one. Le Clezio *et al*. investigate the profile of the out-of-plane component of the total displacement field away from the resonant edge of an aluminum plate.[12]

## III. EXPERIMENTAL RESULTS

Several measurements were performed on different material plates to explore the domain of existence of local resonances.

### A. Local resonance measurements

Laser ultrasonic techniques were used in order to avoid any damping of the plate resonances due to a mechanical contact. In the first experimental setup shown in Fig. 4, the source and detection points are superimposed. The thermoelastic source is a Q-switched Nd:YAG laser (optical wavelength 1064 nm) providing a pulse having 20 ns duration and 4 mJ of energy. The spot diameter of the unfocused beam is equal to 1 mm, in order to efficiently generate the first ZGV mode.[14] Lamb waves were detected by a heterodyne interferometer equipped with a frequency doubled Nd:YAG laser (wavelength 532 nm, power 100 mW). This interferometer is sensitive to any phase shift along the path of the optical probe beam reflected by the moving surface.[18] The normal displacement is recorded using an oscilloscope linked to a computer, which permits to process the data.

Three samples were used in the following experiments: the first one is a 150-mm square Duralumin plate of thickness 1.51 mm, the second one is a 49-mm square crown glass plate of thickness 1.57 mm and the third one is a $15 \times 20$ mm fused silica plate of thickness 1.09 mm.

The first measurements were performed on the 150-mm square Duralumin plate. Material parameters, determined from the ZGV resonance method,[15] were found to be

$$V_L = 6361 \text{ m/s}, \quad V_T = 3134 \text{ m/s}, \quad \nu = 0.3397,$$

with a relative error smaller than 0.1%.

The normal displacements measured when the source and detection points are superimposed in the middle and on the edge of the plate are displayed in Figs. 5(a) and 5(b), respectively. The upper trace corresponds to the ZGV resonance of the $S_1$ Lamb mode, the second ones to the edge resonance. Whereas the amplitudes are comparable, the time decays of the two signals are different. Values of the measured and predicted resonance frequencies and also of the quality factors are given in Table I. The theoretical quality factor is defined as $Q = |\text{Re}[\Omega]/2\text{Im}[\Omega]|$ and the experimental value is deduced from the time decay of the normal displacement. With a difference as small as 0.2%, the agreement between theoretical and experimental resonance frequencies is very good. The rapid decrease of the displacement at the plate edge [Fig. 5(b)] is linked to the imaginary part of the edge resonance frequency. For Duralumin ($\nu \approx 0.34$) the values calculated with Eq. (9) in Ref. 17:

$$\text{Re}[\Omega] = 2.367 \quad \text{and} \quad \text{Im}[\Omega] = -3.35 \times 10^{-3},$$

lead to a theoretical quality factor $Q_{th} = 350$. The experimental value ($Q_{\text{meas}} = 250$) is smaller than the theoretical one. This difference may be ascribed to small geometrical irregularities in the plate edge which broaden the resonance peak. In practice, the quality factor ($Q = 4200$) of the ZGV resonance is limited by local variations in the plate thickness.

Similar measurements were performed for crown glass and fused silica plates. Material parameters of the crown glass were found to be: $V_L = 5744$ m/s, $V_T = 3432$ m/s and $\nu = 0.2224$. This Poisson's ratio is very close to the second value $V_2 = 0.2248$ for which the resonance frequency becomes

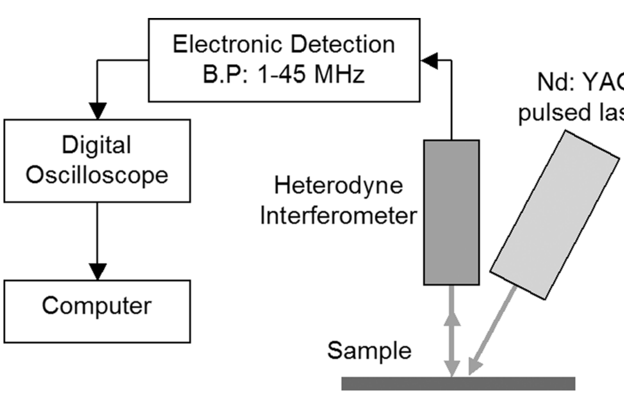


FIG. 4. Experimental setup.

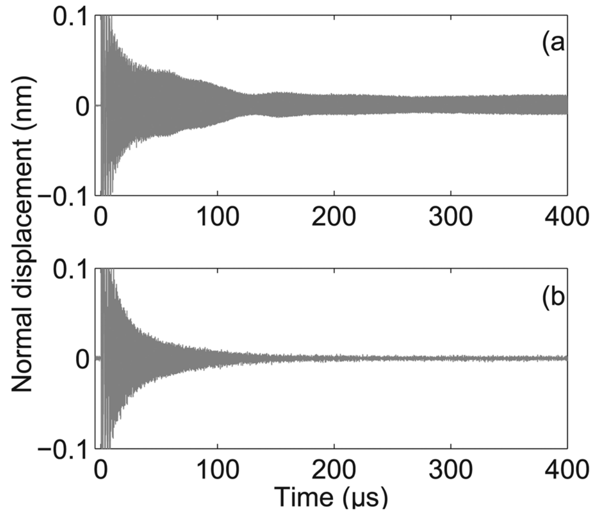


FIG. 5. Normal displacements measured when the source and detection points are superimposed (a) in the middle, (b) near the edge of the Duralumin plate.

TABLE I. Frequencies and quality factors of ZGV and edge resonances, measured and predicted for the three samples of thickness *d*.

| Material | Resonance | $f_{\text{pred}}$ (MHz) | $f_{\text{meas}}$ (MHz) | $Q_{\text{meas}}$ |
|---|---|---|---|---|
| Duralumin | ZGV | 1.905 | 1.901 | 4200 |
| ($d = 1.51$ mm) | Edge | 1.568 | 1.566 | 250 |
| Crown glass | ZGV | 1.738 | 1.738 | 1000 |
| ($d = 1.57$ mm) | Edge | 1.544 | 1.545 | 650 |
| Fused silica | ZGV | 2.640 | 2.626 | 590 |
| ($d = 1.09$ mm) | Edge | 2.389 | 2.378 | 330 |

real. Assuming no material damping, the imaginary part found by Pagneux[17] is very small compared to the real part:

$$\text{Re}[\Omega] = 2.219 \quad \text{and} \quad \text{Im}[\Omega] = -0.45 \times 10^{-6},$$

and the theoretical quality factor of the edge resonance is larger than $10^6$. In practice, this feature is limited by the signal to noise ratio, the material damping and the coupling with the air. Nevertheless, the measured quality factor for the edge resonance: $Q_{\text{meas}} = 650$, is higher than in Duralumin which is in agreement with theoretical predictions.

For the fused silica sample, the parameters were found to be: $V_L = 6032$ m/s, $V_T = 3792$ m/s, and $\nu = 0.1733$. For this plate the quality factor of the ZGV resonance is lower than expected which can be ascribed to a defect in interface parallelism. The quality factor of the edge mode is in between those of Duralumin and crown glass which corresponds to the theory.

## B. Dependence of ZGV and edge resonances on the excitation location

In order to investigate the domain of existence of these resonances, the distance of the source and probe beams from the plate edge was varied in 50 $\mu$m steps on the Duralumin plate. At each distance, the normal displacement $u(r, t)$ was recorded during 390 $\mu$s with a 25 MHz sampling frequency. The time domain Fourier transform $U(r, f)$ was then calculated and is displayed in Fig. 6 between 1.4 MHz and 2.1 MHz. Two horizontal lines can be observed on the frequency B-scan: the higher one corresponds to the $S_1S_2$-ZGV resonance frequency. The lower one, localized near the plate edge, corresponds to the edge resonance and perfectly matches the value calculated from Eq. (1). A slow alteration of the ZGV resonance frequency could be expected while coming toward the edge. However, it is not the case. The amplitude of the ZGV resonance vanishes at a distance from the edge equal to about the plate thickness. The transition

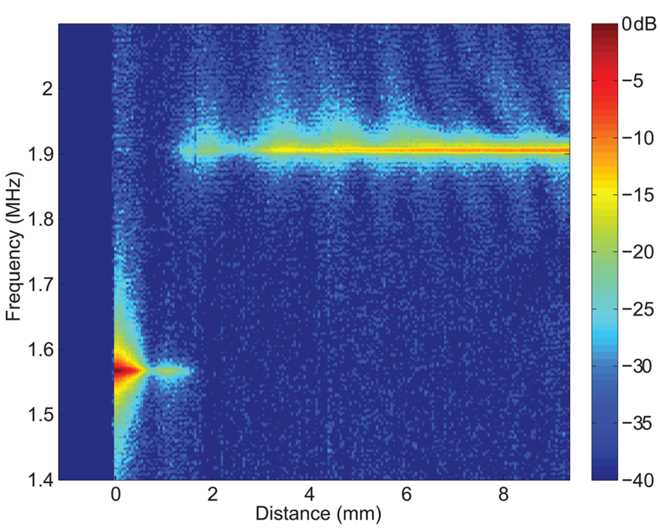


FIG. 6. (Color online) Spatial distribution of the displacement spectrum for the Duralumin plate (dB scale).

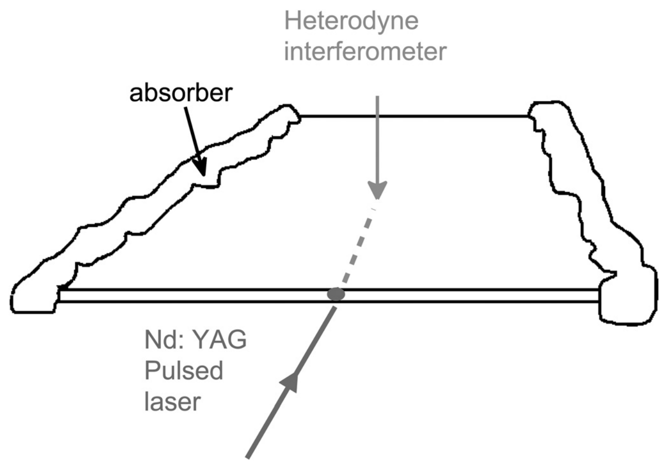


FIG. 7. Experimental setup for the excitation on the edge of the plate.

between ZGV and edge modes is abrupt: the two modes do not exist simultaneously.

In order to measure the profiles of the edge mode, the experimental setup has been modified. The source laser beam was flipped by 90° to hit the edge of the sample, while the interferometer still measures the normal displacement on the plate surface, along a line perpendicular to the edge (Fig. 7). The excitation is then symmetrical and a more precise measurement can be achieved close to the edge since it is not affected by the source thermal signal due to the heating of the air by the laser pulse.

According to the scheme in Fig. 8(a), the laser pulse produces a direct excitation of the edge resonance which lasts

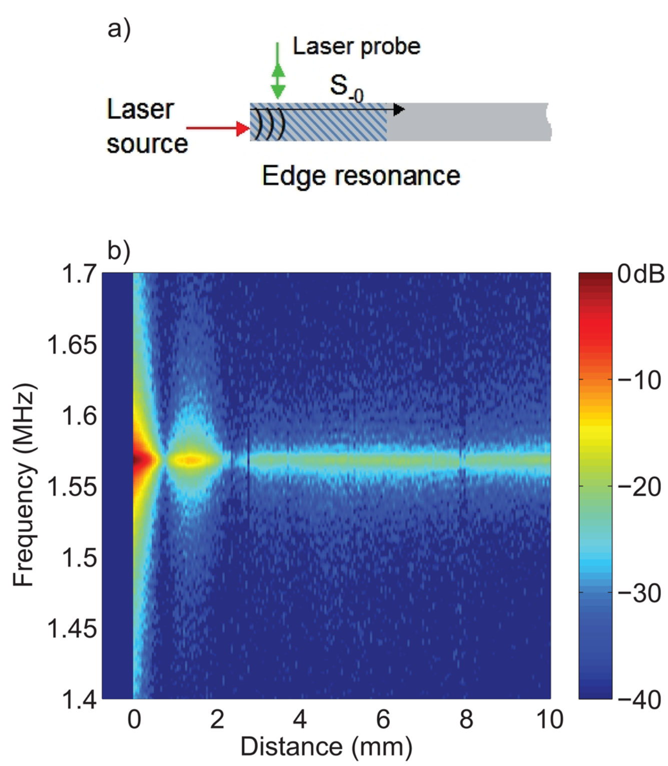


FIG. 8. (Color online) (a) Direct excitation by the laser pulse. (b) Resonance spectrum versus the distance from the edge of the Duralumin plate (dB scale).

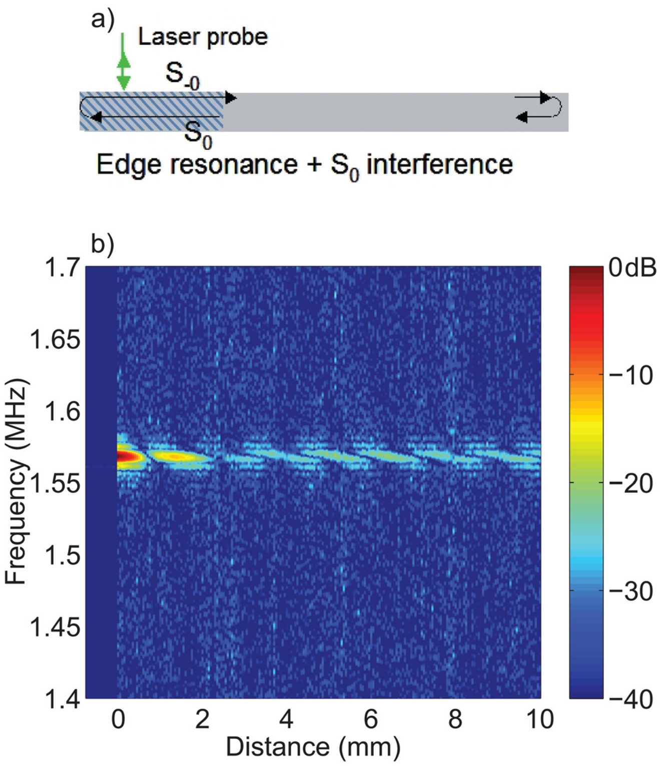


FIG. 9. (Color online) (a) Second excitation by the backward $S_0$ mode. (b) Edge resonance spectrum on the Duralumin plate (dB scale).

about 130 $\mu$s. A transient $S_{-0}$ Lamb mode propagating toward the opposite edge of the plate is also generated. This $S_{-0}$ notation follows the axis orientation chosen by Le Clezio *et al.*[12] The amplitude of the mechanical displacement versus frequency and propagation distance is calculated by a temporal Fourier transform for this time duration and displayed in Fig. 8(b). Close to the edge, the profile presents two nodes corresponding to the interference of the $S_{-0}$ mode with the evanescent modes. At a distance of 4 mm from the edge, evanescent modes vanished and the level remains constant corresponding to the $S_{-0}$ propagating mode which is fed by the resonance. This phenomenon was described by Gazis and Mindlin as follows: "the energy that is temporarily trapped at the edge will gradually leak into the interior of the plate as an extensional wave."[10]

Then, the resonance is excited a second time by the reflection of the $S_{-0}$ mode at the opposite edge of the plate according to the scheme in Fig. 9(a). This reflected $S_0$ mode comes back to the observed region after 150 $\mu$s, and simultaneously excites the edge resonance and interferes with the $S_{-0}$ mode. Indeed, the plate is large enough and the resonance quality factor small enough to ensure that the first resonance vanishes before the $S_0$ reflected mode reaches the first edge leading to a second resonance. This second excitation is similar to the excitation made with a transducer placed at the opposite edge, as in experimental studies by Le Clezio *et al.* for plates[12] or Ratassepp *et al.* for pipes.[13] Thus, it is interesting to analyze the profile in order to compare with theoretical predictions.

The amplitude profiles were calculated versus the frequency for the second time window from 160 to 400 $\mu$s. In

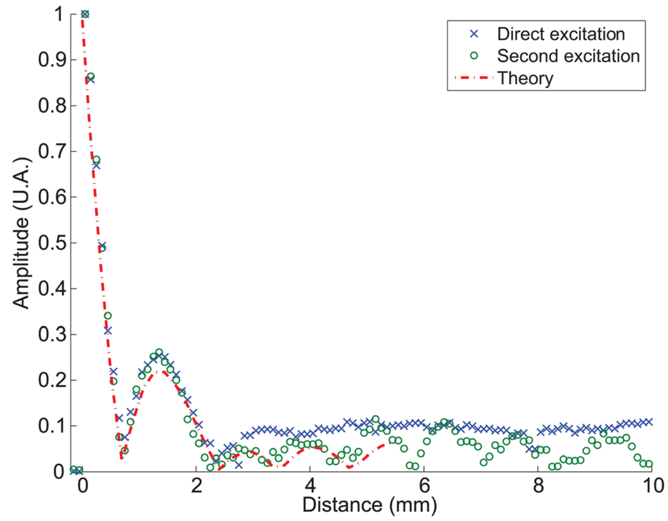


FIG. 10. (Color online) Edge resonance profile: direct excitation (x), second excitation (o) compared to theoretical prediction by Le Clezio *et al.* (dash dotted line).

this case, after the first two nodes, other regularly spaced nodes appear that are due to the interferences between incident and reflected modes $S_0$ and $S_{-0}$.

In Fig. 10, the two profiles obtained at the resonance frequency are compared to the theoretical one calculated by Le Clezio *et al.* for $\nu = 0.33$.[12] The profile obtained with $S_0$ excitation (o) agrees quite well with the theory (dashed dot). The profile obtained by the pulse excitation ($\times$) agrees near the edge, while further, as expected, there is no interference pattern, but the $S_0$ level at the maxima of the former interferences.

In order to confirm which propagating modes contribute to the edge resonance, we have calculated the spatial spectrum by Fourier transform of the field measured over 10 mm. In Fig. 11, the amplitude of the mechanical displacement at the edge resonance frequency is plotted for real wave number. For the first time window (solid line), it is clear that the $S_{-0}$ mode dominates all other modes at this frequency, this

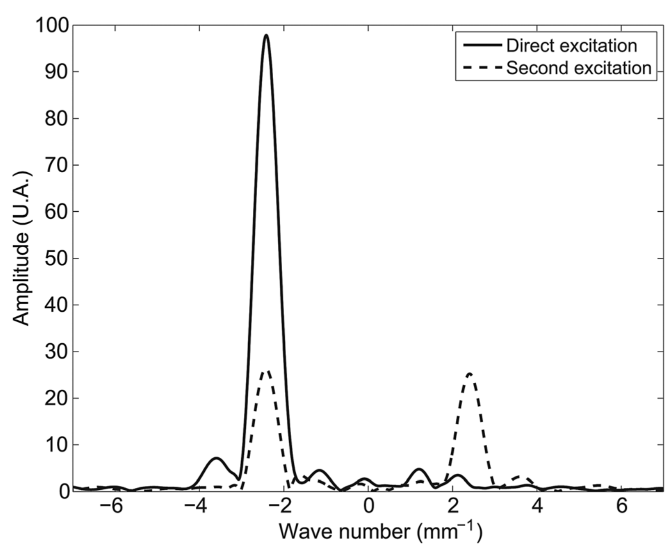


FIG. 11. Spatial Fourier transform of the normal displacement at the edge resonance frequency for the direct laser pulse excitation (solid line) and for the second excitation by the backward $S_0$ mode (dashed line).

confirms the fact that the edge resonance feeds the propagating mode. This behavior differs from those of the ZGV modes that are only associated to non propagating modes. It is coherent with the relatively low quality factor of the edge mode. For the $S_0$ excitation, i.e., the second time window (dashed line), the contribution of the incident $S_0$ and reflected $S_{-0}$ symmetric Lamb modes dominates with similar amplitude. This analysis confirms that $S_0$ is the only propagating mode that contributes to the edge resonance.

## IV. CONCLUSION

Zero group velocity (ZGV) Lamb modes and edge resonance of an isotropic plate were experimentally investigated using laser ultrasonic techniques. It has been shown that ZGV modes can be observed without any frequency change at a distance as close as the plate thickness from the edge. Coming toward the edge, the amplitude of the ZGV resonance vanishes whereas the edge mode resonance appears at a lower frequency. The transition between the generation of ZGV and edge modes is abrupt. Resonance frequencies measured for Duralumin, crown glass and fused silica plates are in a very good agreement with the theoretical values, the frequency dependence on Poisson's ratio was validated for these different materials. Moreover, using a laser generation on the plate edge, the edge resonance was excited solely and the spatial distribution of the mode measured on the plate face by scanning the laser probe. The experimental profile was found to be in a good agreement with the theoretical one given in the literature.


[1]L. Rayleigh, "On the free vibrations of an infinite plate of homogeneous isotropic elastic matter," Proc. London Math. Soc. **20**, 225–234 (1889).

[2]R. D. Mindlin, *An Introduction to the Mathematical Theory of Vibrations of Elastic Plates*, US army Signal Corps Eng. Lab., Ft Monmouth, NJ, Monograph. Sec. 2.11 (1955), edited by J. Yang (World Scientific, Singapore, 2006).

[3]I. Tolstoy and E. Usdin, "Wave propagation in elastic plates: Low and high mode dispersion," J. Acoust. Soc. Am. **29**, 37–42 (1957).

[4]A. Gibson and J. S. Popovics, "Lamb wave basis for impact-echo method analysis," J. Eng. Mech. **131**(4), pp 438–443 (2005).

[5]C. Prada, O. Balogun, and T. W. Murray, "Laser based ultrasonic generation and detection of zero-group velocity Lamb waves in thin plates," Appl. Phys. Lett. **87**, 194109 (2005).

[6]E. A. G. Shaw, "On the resonant vibrations of thick barium titanate disks," J. Acoust. Soc. Am. **28**, 38–50 (1956).

[7]J. Oliver, "Elastic wave dispersion curves in a cylindrical rod by a wide-band short- duration pulse technique," J. Acoust. Soc. Am. **29**, 189–194 (1957).

[8]M. Onoe, "Contour vibrations of thin rectangular plates," J. Acoust. Soc. Am. **30**, 1159–1162 (1958).

[9]R. D. Mindlin and M. A. Medick, "Extensional vibrations of plates," J. Appl. Mech. **26**, 561–569 (1959).

[10]D. C. Gazis and R. D. Mindlin, "Extensional vibrations and waves in a circular disk and a semi-infinite plate," J. Appl. Mech. **27**, 541–547 (1960).

[11]H. D. McNiven, "Extensional waves in a semi-infinite elastic rod," J. Acoust. Soc. Am. **33**, 23–27 (1961).

[12]E. Le Clezio, M. V. Predoi, M. Castaings, B. Hosten, and M. Rousseau, "Numerical predictions and experiments on the free-plate edge mode," Ultrasonics, **41**, 25–40 (2003).

[13]M. Ratassepp, A. Klauson, F. Chati, F. Léon, and G. Maze, "Edge resonance in semi-infinite thick pipe: Numerical predictions and measurements," J. Acoust. Soc. Am. **124**(2), 875–885 (2008).

[14]D. Clorennec, C. Prada, D. Royer, and T. W. Murray, "Laser impulse generation and interferometer detection of zero-group velocity Lamb modes," Appl. Phys. Lett. **89**, 024101 (2006).

[15]D. Clorennec, C. Prada, and D. Royer, "Local and noncontact measurement of bulk acoustic wave velocities in thin isotropic plates and shells using zero-group velocity Lamb modes," J. Appl. Phys. **101**, 034908 (2007).

[16]C. Prada, D. Clorennec, and D. Royer, "Local vibration of an elastic plate and zero-group velocity Lamb modes," J. Acoust. Soc. Am. **124**, 203–212(2008).

[17]V. Pagneux, "Revisiting the edge resonance for Lamb waves in a semi-infinite plate," J. Acoust. Soc. Am **120**, 649–653 (2006).

[18]D. Royer and E. Dieulesaint, *Proceedings of the 1986 IEEE Ultrasonics Symposium* (IEEE, New York, 1986) p. 527.