Universal statistics of waves in a random time-varying medium
Supplemental Material

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I. RANDOM BARRIERS MODEL

In this section we describe a model for a time-disordered homogeneous medium that could be used as an alternative to the δ-kicks model presented in the main text. $\Omega^2(t)$ is considered to be a chain of rectangular barriers, as represented in Fig. 1. Each barrier is denoted as a kick, with the times t'_j and t_j defining the onset and end of kick number j. The times t'_j and t_j, as well as the kick strengths (barrier heights) Ω_j^2 are random variables.

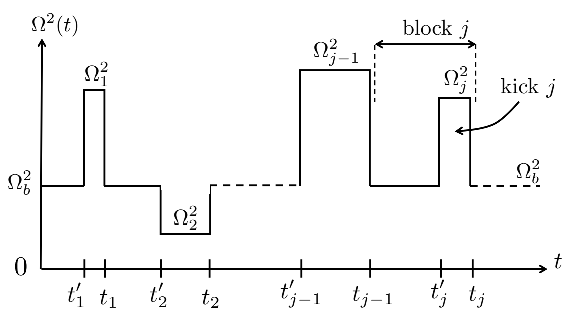

Figure 1. Random chain of rectangular barriers, each of them referred to as a "kick". The barrier heights Ω_j^2 and the times t'_j and t_j are random variables. After each kick the medium recovers the background value Ω_b^2.

The transfer matrix $\mathbf{M}_j$ of a single block j takes the form

$$\mathbf{M}_j = \begin{bmatrix} a_j & b_j \\ b_j^* & a_j^* \end{bmatrix} \tag{1}$$

with

$$a_j = T_j T'_j \exp[-i(\theta_j + \phi_j)] + R_j R'_j \exp[-i(\theta_j - \phi_j)] \tag{2}$$

$$b_j = T_j R'_j \exp[i(\theta_j - \phi_j)] + R_j T'_j \exp[i(\theta_j + \phi_j)]\,. \tag{3}$$

Here, $T_j = (\Omega_b + \Omega_j)/(2\Omega_b)$, $T'_j = (\Omega_b + \Omega_j)/(2\Omega_j)$, $R_j = (\Omega_b - \Omega_j)/(2\Omega_b)$ and $R'_j = (\Omega_j - \Omega_b)/(2\Omega_j)$ are time-domain transmission and reflection coefficients, that can be deduced from the continuity of the field and its time derivative at the times $t = t'_j$ and $t = t_j$. The phases $\phi_j = \Omega_j(t_j - t'_j)$ and $\theta_j = \Omega_b(t'_j - t_{j-1})$ correspond to free propagation during the kick, and free propagation between two succesive kicks, respectively.

We note that the transfer matrix takes the same form as Eq. (6) in the main text, but with different matrix elements. It can be verified that the matrix elements also satisfy Eqs. (8) and (9) of the main text, which are very general and independent of the model of disorder (see Ref. [1] for a general derivation of these properties).

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II. TRANSFER MATRIX FOR δ-KICKS

In this section we deduce the expression of the transfer matrix for the δ-kicks model described in Fig. 1 of the main text, starting from the random barriers model decribed in the previous section. For kick number j we take a barrier with duration δt and amplitude such that

$$t_j - t'_j = \delta t, \quad \Omega_j^2 - \Omega_b^2 = v_j/\delta t. \tag{4}$$

Taking the limit $\delta t \to 0$ leads to $\Omega^2(t) = \Omega_b^2 + \sum_j v_j \delta(t - t_j)$ which corresponds to the δ-kicks model with v_j the amplitude of kick number j. Taking the same limit for the coefficients of the transfer matrix $\mathbf{M}_j$ in Eqs. (2) and (3) leads to

$$a_j = [1 - i v_j/(2\Omega_b)] \exp(-i\theta_j), \tag{5}$$

$$b_j = -i v_j/(2\Omega_b) \exp(i\theta_j). \tag{6}$$

These expressions correspond to Eq. (7) of the main text.

III. DERIVATION OF THE MOMENT RELATION LEADING TO EQ. (20)

In this section we derive the moment relation $\langle z_j^n \rangle = (n!) \langle z_j \rangle^n$ that is used as a step in the derivation of Eq. (20) in the main text. We start by raising Eq. (19) of the main text to the power n. Keeping terms of order $\sqrt{\beta_j}$ and β_j, we obtain

$$z_j^n = z_{j-1}^n + n\beta_j z_{j-1}^{n-1} + 2n z_{j-1}^{n-1} \sqrt{\beta_j z_{j-1}} \cos(\Theta_j) + 2n(n-1) z_{j-1}^{n-1} \beta_j \cos(\Theta_j)^2 + \mathrm{O}(\beta_j^{3/2}). \tag{7}$$

We perform an average over Θ, assuming that it is fully randomized after a sufficiently large number of kicks (this hypothesis is discussed in the main text) and independent of β_j, and a subsequent average over an arbitrary distribution for β_j. This leads to the following recursion relation for the moments of z

$$\langle z_j^n \rangle = \langle z_{j-1}^n \rangle + n^2 \langle \beta \rangle \langle z_{j-1}^{n-1} \rangle \,, \tag{8}$$

with the first moment given by $\langle z_j \rangle = j \langle \beta \rangle$. We now assume that the moment of order n is

$$\langle z_j^n \rangle = n! \langle z_j \rangle^n = n! j^n \langle \beta \rangle^n . \tag{9}$$

Inserting the above relation in Eq. (8), it is easy to see that it is satisfied up to terms of order $1/j^2$. This concludes the derivation of the relation $\langle z_j^n \rangle = n! \langle z_j \rangle^n$, valid for large enough j.

IV. DERIVATION OF MELNIKOV'S EQUATION

In this section we derive Eq. (21) of the main text. The derivation can be found in Ref. [1], and we summarize the main steps here. We starts with the basic recursion relation for the variable z [Eq. (12) in the main text], that can be rewritten as

$$z_j = z_{j-1} + \beta_j (1 + 2 z_{j-1}) + 2 \sqrt{\beta_j (1+\beta_j) z_{j-1} (1 + z_{j-1})} \cos \Theta_j . \tag{10}$$

Considering as above that Θ_j is fully randomized, the above relation can be transformed into a recursion relation for the probability density $P_j(z)$, which takes the form

$$P_j(z) = \int_0^\infty f(\beta) d\beta \int_0^{2\pi} \frac{d\Theta}{2\pi} \times P_{j-1} \left(z + \beta(1+2z) - 2\sqrt{\beta(1+\beta) z (1+z)} \cos\Theta \right) , \tag{11}$$

where $f(\beta)$ is the probability density asociated to the random variable β. For weak disorder such that $\beta \ll 1$, we can perform a second order Taylor expansion for P_{j-1}, which leads to

$$P_{j-1}(z+\epsilon) = P_{j-1}(z) + \epsilon \frac{\partial P_{j-1}(z)}{\partial z} + \frac{\epsilon^2}{2} \frac{\partial^2 P_{j-1}(z)}{\partial z^2} + \mathrm{O}(\epsilon^3) \,, \tag{12}$$

where $\epsilon = \beta(1+2z) - 2\sqrt{\beta(1+\beta)z(1+z)} \cos\Theta$. Substituting in Eq. (11), we obtain

$$P_j(z) = P_{j-1}(z) + \langle \beta \rangle (1+2z) \frac{\partial P_{j-1}(z)}{\partial z} + \langle \beta \rangle z (1+z) \frac{\partial^2 P_{j-1}(z)}{\partial z^2} + \mathrm{O}(\beta^2) \,. \tag{13}$$

This can be factorized in the form

$$P_j(z) = P_{j-1}(z) + \langle \beta \rangle \frac{\partial}{\partial z} \left[(z + z^2) \frac{\partial P_{j-1}}{\partial z}(z) \right] \,, \tag{14}$$

where terms beyond first order have been neglected. This is Eq. (21) in the main text, which in the continuous limit leads to Melnikov's equation (22).

[1] P.A. Mello and N. Kumar, *Quantum Transport in Mesoscopic Systems: Complexity and Statistical Fluctuations. A Maximum Entropy Viewpoint* (Oxford University Press, 2004).